Lecture 4 - January 19

Math Review

Predicate Logic
Sets

Announcement

- Lab1 released
  - + tutorial videos
  - + problems to solve
  - + Study along with the Math Review lecture notes.

Predicate Logic: Quantifiers

- syntax
- base cases in programming

$$\forall i \bullet R(i) \Rightarrow P(i)$$

for empty array R(i) always false

universal property

false ⇒ T ≡ T

$$\exists i \bullet R(i) \land P(i)$$

for empty array, R(i) false

existential property

false ∧ F ≡ F

universe of disclosure

↳ what if it's empty

i.e.. R(i) false for any possible value of i

```
boolean allPositive(int[] a){
  if(a.length == 0){ ① return true; }
}
```

∵ no witness in empty array can prove otherwise

```
boolean somePositive(int[] a){
  if(a.length == 0){ ② return false; }
}
```

∵ no witness in empty array can prove so

$\subseteq$

$\subset$

<u>Exercises</u> $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{Z} \subset \mathbb{N}$

$\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{N}$

$\mathbb{N}$ the set of <u>all</u> natural #s

$(0, 1, 2, \ldots, +\infty)$

$\mathbb{Z}$ the set of all integer #s

$(-\infty, \ldots, 0, \ldots, +\infty)$

$$\forall i, j \cdot i \in \mathbb{N} \wedge j \in \mathbb{Z} \Rightarrow P(i, j)$$

↳ should hold for all combinations of i, j.

↳ pay attention to how $\forall$ and $\exists$ should be written in Rodin.

Logical Quantifiers: Examples

$\forall i \bullet i \in \mathbb{N} \Rightarrow i \geq \underline{\underline{0}}$ T

0, 1, 2, ...

$\forall i \bullet i \in \underline{\underline{\mathbb{Z}}} \Rightarrow i \geq 0$ F

$-2 \in$

F

$\forall \underline{i, j} \bullet i \in \mathbb{Z} \land j \in \mathbb{Z} \Rightarrow \underline{i < j \lor i > j}$ → $i = j$ $\underline{\underline{(3, 3)}}$

$3 \in \mathbb{Z} \land 3 \in \mathbb{Z}$

$3 < 3 \lor 3 < 3$

F

$\exists i \bullet i \in \mathbb{N} \land i \geq 0$ T

T e.g. witness: 0, 1, ...

✓

$\exists i \bullet i \in \mathbb{Z} \land i \geq 0$ T e.g. 0

$3 < 4$ $3 > 4$

$\exists i, j \bullet \boxed{i \in \mathbb{Z} \land j \in \mathbb{Z}} \land \boxed{(i < j \lor i > j)}$ T

$3 \in \mathbb{Z} \land 4 \in \mathbb{Z}$ T ↳ witness: $i = 3$, $j = 4$ T

Logical Quantifiers: Examples

How to prove $\forall i \bullet R(i) \Rightarrow P(i)$?

harder

(1) show $\neg R(\bar{i})$ (i.e. empty universe of disclosure)

zero of $\Rightarrow$: $F \Rightarrow P \equiv \boxed{T}$.

(2) show $R(\bar{i})$, $P(\bar{i})$ (i.e. all elements in non-empty array).

$T \Rightarrow T \equiv \boxed{T}$

How to prove $\exists i \bullet R(i) \land P(i)$?

similar

(1) show a witness $\bar{i}$ s.t. $R(\bar{i})$, $P(\bar{i})$

How to disprove $\forall i \bullet R(i) \Rightarrow P(i)$?

(1) give a counter-example/witness $\underline{\underline{\bar{i}}}$ s.t. $R(\bar{i})$, $\neg P(\bar{i})$

(i.e., an element in array but does not satisfy property).

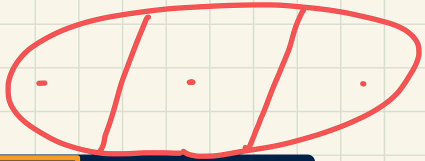
How to disprove $\exists i \bullet R(i) \land P(i)$?

harder

(1) show $\neg R(\bar{i})$ (empty). $F \land P \equiv \underline{F}$

(2) show $R(\bar{i})$, $\neg P(\bar{i})$

Prove/Disprove Logical Quantifications



- Prove or disprove: $\forall x \bullet \boxed{(x \in \mathbb{Z} \wedge 1 \leq x \leq 10)} \Rightarrow \underline{x > 0}$.

non-empty : 1, 2, 3, ..., 10 ⇒ all > 0.

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 1$.

↳ counter-example/witness: $x = 1$

$T \Rightarrow F \equiv$ Ⓕ.

- Prove or disprove: $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 1$.

↳ witness: 2 $\quad T \wedge T \equiv$ Ⓣ.

- Prove or disprove that $\exists x \bullet \underline{(x \in \mathbb{Z} \wedge 1 \leq x \leq 10)} \wedge x > 10$?

↳ non-empty: 1, 2, 3, ... 10

↳ all make $x > 10$ Ⓕ.

Logical Quantifications: Conversions

$R(x)$: $x \in$ 3342_class

$P(x)$: x receives A+

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\underline{(\forall X \bullet R(X) \Rightarrow P(X))} \Leftrightarrow \neg(\exists X \bullet R \wedge \neg P)$$

$$\equiv \neg(\exists x \bullet \neg(R(x) \Rightarrow P(x)))$$

$$\equiv \neg(\exists x \bullet \underline{\neg}(\underline{\neg} R(x) \vee P(x))) \equiv \neg(\exists x \bullet \underline{R(x) \wedge \neg P(x)})$$

$$(\exists X \bullet R \wedge P) \Leftrightarrow \neg(\forall X \bullet R \Rightarrow \neg P)$$

Exercise!..

✓ **De Morgan**

Lecture 1b

Review on Math: Sets

$$\{1,2,3\} = \{2,3,1\}$$

Empty Set: $\emptyset$

$$\{\}$$

$$\{1, \underline{2}, 3, \underline{2}\} \quad X$$

$$|\emptyset| = 0$$

$$|\{1,2,3\}| = 3$$

Set Comprehension

$$\{x \mid x \in \mathbb{N} \wedge x > 5\}$$

$$=$$

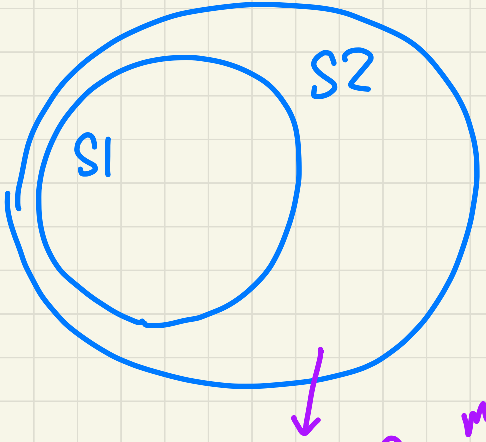
$$\{6, 7, 8, \dots \infty\}$$

$$5 \in \mathbb{N}$$

$$-2 \notin \mathbb{N}$$

$$\Big\{ \underbrace{\qquad\qquad}_{\text{list of variables}} \Big| \underbrace{\qquad\qquad\qquad\qquad}_{\text{constraint}} \Big\}$$

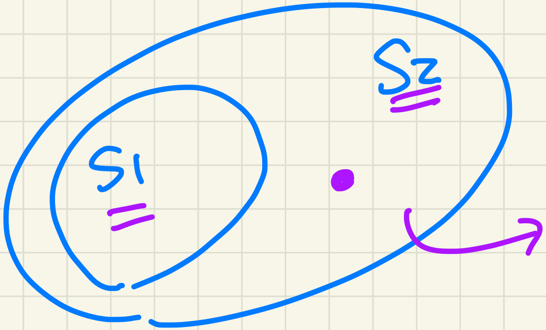
$S_1 \subseteq S_2$



$S_2 \setminus S_1$ may be ϕ

in S_2 but not in S_1

$S_1 \subset S_2$



$S_2 \setminus S_1$ must be non-empty

Sets: Exercises

Set membership: Rewrite $e \notin S$ in terms of $\in$ and $\neg$

Find a common pattern for defining:

1. = (numerical equality) via $\leq$ and $\geq$
2. = (set equality) via $\subseteq$ and $\supseteq$

$S = \{1, 2, 3\}$, $T = \{2, 3, 1\}$, $U = \{3, 2\}$

	S		T		U	
S	$\subseteq$	$\subset$	$\subseteq$	$\subset$	$\subseteq$	$\subset$
T	$\subseteq$	$\subset$	$\subseteq$	$\subset$	$\subseteq$	$\subset$
U	$\subseteq$	$\subset$	$\subseteq$	$\subset$	$\subseteq$	$\subset$

Is set difference (\) commutative?

Power Set

$$\mathbb{P}(S) = \{ x \mid x \subseteq S \}$$

each member in a power set is a subset.

Calculate the power set of {1, 2, 3}.

$$\mathbb{P}(\{1,2,3\}) = \left\{ \begin{array}{ll} \emptyset, & \text{/* card is 0 */} \\ \{1\}, \{2\}, \{3\}, & \text{how many: } \binom{3}{1} = 3 \\ \{1,2\}, \{2,3\}, \{1,3\}, & \text{how many: } \binom{3}{2} = 3 \\ \{1,2,3\} & \text{/* card } |\{1,2,3\}| \end{array} \right\}$$

subsets of card. 1

subsets of card. 2

Given a set S, formulate the cardinality of its power set.

$$\underbrace{\binom{|S|}{0}}_{\emptyset} + \underbrace{\binom{|S|}{1}}_{\text{subsets of card 1}} + \cdots + \underbrace{\binom{|S|}{|S|}}_{S} = \sum_{c=0}^{|S|} \binom{|S|}{c}$$

$$\binom{n}{r} = \frac{n!}{(n-r)!\ r!}$$

$$\binom{n}{r} = \binom{n}{n-r}$$